Note on homework:

13.6: Assume that f is a nonzero hdom fcn on $\mathbb{C}$.

Find the Taylor series of

$g(z) = \dfrac{3z}{4-iz}$ centered at $z = 1+i$.

For which z does it converge?

Does the Taylor series converge to the function?

Want: $\dfrac{3z}{4-iz} = \sum_{k \geq 0} c_k (z-(1+i))^k$

Solution:
$$\frac{3z}{4-iz} = \frac{3(z-(1+i)+(1+i))}{4 - i(z-(1+i)+(1+i))}$$

$$= \frac{3(1+i) + 3(z-(1+i))}{5-i - i(z-(1+i))}$$

$$= \left[\frac{3+3i+3(z-(1+i))}{(5-i)}\right] \cdot \frac{1}{\left(1 - \frac{i(z-(1+i))}{(5-i)}\right)}$$

$$= \left[\frac{(3+3i) + 3(z-(1+i))}{(5-i)}\right] \sum_{k\geq 0} \left(\frac{i(z-(1+i))}{(5-i)}\right)^k$$

$$\left(\frac{i}{5-i}\right)^k (z-(1+i))^k$$

$$= \sum_{k\geq 0} C_k \, (z-(1+i))^k$$

where $C_k = \begin{cases} \left(\frac{3+3i}{5-i}\right)\left(\frac{i}{5-i}\right)^k + \left(\frac{3}{5-i}\right)\left(\frac{i}{5-i}\right)^{k-1} & \text{if } k \geq 0 \\ \left(\frac{3+3i}{5-i}\right) & \text{if } k = 0 \end{cases}$

The $(z-(1+i))^0$ term will only be $\left(\frac{3+3i}{5-i}\right) \cdot 1$

$$(A+Bw)\sum D_k w^k$$

$$= \sum_{k\geq 0} AD_k w^k + \sum_{k\geq 0} BD_k w^{k+1}$$

$$= \sum_{k\geq 1} (AD_k + BD_{k-1}) w^k + AD_0$$

$$AD_0 + AD_1 w^1 + AD_2 w^2 + AD_3 w^3$$
$$+ BD_0 w^1 + BD_1 w^2 + BD_2 w^3 + \cdots$$

This series converges iff $\left| \frac{i(z-(1+i))}{5-i} \right| < 1$

$$\iff |z-(1+i)| < |5-i| = \sqrt{26}.$$

and it converges exactly to $g(z)$ for those values of z.

<u>Thm</u> Let f be a holomorphic function in a domain D, and let $z_0 \in D$. Let $R>0$ be the largest positive # such that $B(z_0, R) \subseteq D$. (it's allowed that $R=\infty$ if f is entire.)

Then $f(z) = \sum_{k \geq 0} C_k (z-z_0)^k$, and the series converges for $|z-z_0|<R$, diverges for $|z-z_0|>R$, and the series converges to $f(z)$ for $|z-z_0|<R$.

<u>and</u> $$C_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\omega)\, d\omega}{(\omega - z_0)^{k+1}};$$

where $C_r(z_0) = \{z : |z-z_0| = r\}$, $0<r<R$.

Proof:

expand in geometric series centered at z_0

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-(z-z_0+z_0)} = \frac{f(w)}{(w-z_0)-(z-z_0)}$$

$$= \frac{f(w)}{(w-z_0)} \cdot \frac{1}{\left(1-\left(\frac{z-z_0}{w-z_0}\right)\right)}$$

$$= \frac{f(w)}{w-z_0} \sum \left(\frac{z-z_0}{w-z_0}\right)^k$$

converges to $\frac{f(w)}{w-z}$ for $\left|\frac{z-z_0}{w-z_0}\right| < 1$

$\Longleftrightarrow$ converges to $\frac{f(w)}{w-z} \Longleftrightarrow |z-z_0| < |w-z_0|$.

Let's use CIF:

$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z}\,dw \quad \text{for } |z-z_0| < r$$

$$\overset{\text{by above}}{=} \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z_0} \sum_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k dw$$

works since. $|z-z_0| < r = |w-z_0|$ ✓

Can switch $\int \sum_1^\infty f_n = \sum_1^\infty \int f_n$ if $\sum_1^\infty f_n$ converges uniformly to an integrable fcn f.

Note: $\frac{1}{1-z} = \sum_{k\geq 0} z^k$ ← This converges uniformly on $\overline{B(0,\tau)}$ if $0<\tau<1$.

Continuing:

$$f(z) = \sum_{k\geq 0}^\infty \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z_0}\left(\frac{z-z_0}{w-z_0}\right)^k dw$$

$$= \sum_{k\geq 0} \frac{1}{2\pi i}(z-z_0)^k \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{k+1}}\, dw$$

$$= \sum_{k\geq 0}\left(\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{k+1}}\, dw\right)(z-z_0)^k.$$

$\boxed{}$

$$\left[\text{also } \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)\,dw}{(w-z_0)^{k+1}}\right]$$

Note: This formula gives the radius of convergence R of the series!.

Example of an application:

Consider $f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)}$

$$= \sum_{k\geq 0} (-z^2)^k = 1 - z^2 + z^4 - z^6 \dots$$

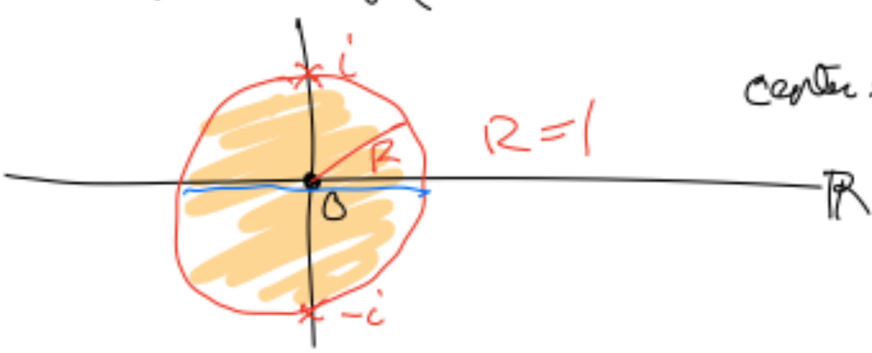
converges iff $|z^2| < 1 \iff |z| < 1$.

$\therefore$

$$\underbrace{\frac{1}{2\pi i}\int_{C_r(0)} \frac{1}{(1+w^2)\,w^{k+1}}\,dw}_{k^{\text{th}} \text{ coefficient of Taylor series}} = \begin{cases} (-1)^n & \text{if } k = 2n \text{ for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\left(\frac{1}{2\pi i}\int_{C_r(0)} \frac{f(w)\,dw}{(w-0)^{k+1}}\right)$$

Domain of $\left(\frac{1}{1+z^2}\right)$: $D = \mathbb{C} - \{i, -i\}$.



center $= z_0 = 0$

Note: This is why the real-valued fcn $f(x) = \frac{1}{1+x^2}$ on $\mathbb{R}$ is a smooth fcn on all of $\mathbb{R}$, but its Taylor series centered at 0 only converges for $|x| < 1$.

$\boxed{\text{Ex}}$ Find the radius of convergence of the Taylor series of $g(z) = \dfrac{e^{z^4} \cdot \cos(z^2 + 2z^5 - z)}{z^2 + 5i}$ centered at $z = 7 - 49i$.

Solution: $g(z)$ is holomorphic except when

$$z^2 = -5i = 5e^{-i\pi/2}$$

$$z = \pm\sqrt{5}\, \underbrace{e^{-i\pi/4}}_{\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)} = \sqrt{\tfrac{5}{2}} - \sqrt{\tfrac{5}{2}}\, i$$

$$\text{or } -\sqrt{\tfrac{5}{2}} + \sqrt{\tfrac{5}{2}}\, i.$$

$$\therefore R = \left| (7-49i) - \left(\sqrt{\tfrac{5}{2}} - \sqrt{\tfrac{5}{2}}\, i\right) \right|$$

$$= \sqrt{\left(7 - \sqrt{\tfrac{5}{2}}\right)^2 + \left(-49 + \sqrt{\tfrac{5}{2}}\right)^2}.$$

Uniform convergence of Taylor series on compact subsets of the open disk of convergence.

We say that $\sum_{n\geq 0} f_n(z)$ converges uniformly to $f(z)$ on a set S if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall k \geq N$

Thank you, Segun!

$$\left| f(z) - \sum_{n=0}^{k} f_n(z) \right| < \varepsilon$$

This N works for every z

$\forall z \in S$.

By contrast: We say $\sum_{n\geq 0} f_n(z)$ converges to $f(z)$ on a set S if $\forall z \in S$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall k \geq N$

this N works for that one specific z.

$$\left| f(z) - \sum_{n=0}^{k} f_n(z) \right| < \varepsilon.$$

Thanks again Segun!

Weierstrass M-Test: If a sequence of functions $f_n(z)$ satisfies $|f_n(z)| \leq M_n$ for all $z \in S$ ($\subseteq$ domain f_n $\forall n$), and if

$\sum_{n=0}^{\infty} M_n$ converges, then

$\sum_{n=0}^{\infty} f_n(z)$ converges uniformly to a function on S.

In the above, if each $f_n(z)$ is holomorphic on S, then $f(z)=\sum_{n=0}^{\infty} f_n(z)$ is holomorphic on S.

<u>Thm:</u> If $f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$ converges to the holom. fcn f on $B(z_0,R)$, then for any r s.t. $0<r<R$, the series converges uniformly to f on $\overline{B(z_0,r)}$.

<u>Lemma</u> If $\sum f_n \to f$ uniformly on a set S, and if $A \subseteq S$, then also $\sum f_n \to f$ uniformly on A.

It turns that a complex-valued fcn $f: \underset{\text{domain}}{D} \to \mathbb{C}$ has a Taylor series that converges on a positive radius to the fcn at each point $\iff$ $f: D \to \mathbb{C}$ is holomorphic.